# Collapsing Non-deterministic Automata

E0 222: ATC - Seminar

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# Introduction

## Isomorphism

NFAs are *not* necessarily unique up to isomorphism.

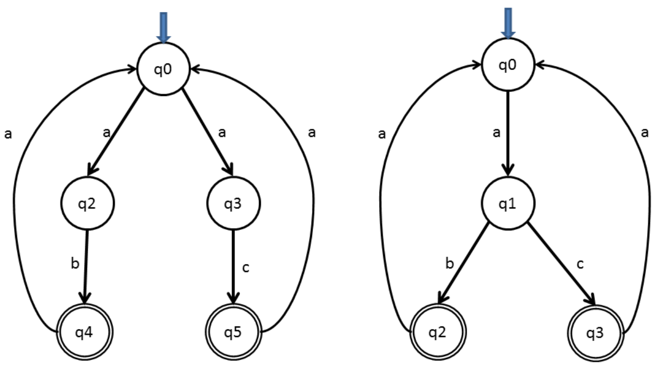



Figure : Two NFAs accepting same language

Both the NFA above cannot be further collapsed, but they are *not* isomorphic. Therefore, collapsing procedures of DFA cannot be applied

# Binary Relation

Consider two NFAs,

$$M = (Q_M, \Sigma, \Delta_M, S_M, F_M)$$
$$N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$$

Then,

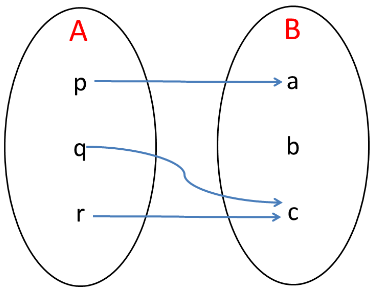
$$\approx \overset{def}{\subseteq} Q_M \times Q_N$$

# Binary Relation

For, $B \subseteq Q_N$,

$$C_{\approx}(B) \stackrel{def}{=} \{p \in Q_M | \exists q \in B, p_{\approx} q\}$$

and , similarly defining $C_{\approx}(A), A \in Q_M)$




(Here C≈(B) = {p,q,r} and C≈(A) = {a,c} from the above definition)

Figure : Binary Relation between two NFA

Extending this relation to the subsets of $Q_M$ and $Q_N$

$$A_{\approx} B \stackrel{def}{\iff} A \subseteq C_{\approx}(B) \; and \; B \subseteq B_{\approx}(A)$$

## Bisimulation

The same relation can be called *Bisimulation* if it satisfies following properties:-

1. $S_M \approx S_N$
2. $If\ p \approx q, then\ \forall a \in \Sigma, \Delta_M(p, a) \approx \Delta_N(q, a)$
3. $If\ p \approx q, then\ p \in F_M \iff q \in F_N$

# Properties of Bisimulation

1. Bisimulation is Symmetric
2. Bisimulation is Transitive
3. Union of nonempty family of bisimulations between $M$ and $N$ is bisimulation between $M$ and $N$

# Bi-similar Automata accepts same set

Let $\approx$ be the bisimulation between $M$ and $N$. Then if $A{\approx}B$ then

$$\forall x \in \Sigma^*, \hat{\Delta}_M(A, x){\approx}\hat{\Delta}_N(B, x)$$

Then we can say,

$$\forall x \in \Sigma^*, \hat{\Delta}_M(A, x) \cap F_M \neq \phi \iff \hat{\Delta}_N(B, x) \cap F_N \neq \phi$$

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- Lemma: A state of $M$ is in support of all bisimulations of $M$ iff it is accessible.
- We can, hence remove all inaccessible states in an NFA

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  - $A \equiv B \Leftrightarrow A' = B'$
  - $A \gtrsim A'$
- Define Quotient automaton

$$M' \stackrel{def}{=} (Q', \Sigma, \Delta', S', F')$$

Where,

$$\Delta'([p], a) \stackrel{def}{=} \Delta(p, a)'$$

## Minimality of $M'$

- The relation $\gtrsim$ is a bisimulation between $M$ and $M'$. So, $M'$ accepts the same set as $M$.
- The only *autobisimulation* on $M'$ is the *identity* relation. So, $M'$ cannot be collapsed further.

## Autobisimulation

**Theorem**

Let $M$ be an $NFA$ with no inaccessible states and let $\equiv_M$ be the maximal autobisimulation on $M$. The quotient automaton $M'$ is the minimal automaton bisimilar to $M$ and is unique upto isomorphism.

# Reference

[1] Dexter C Kozen, *Automata Theory and Computability*, pg 100 - 107.